Creation Operators for 2-branes and Duality in BF and Chern-Simons Theories in D=5

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Abstract

We explicitly construct the creation operators for the quantum field configurations associated to quantum membranes (2-branes) in BF and generalized Chern-Simons theories in a spacetime of dimension D=5. The creation operators for quantum excitations carrying topological charge are also obtained in the same theories. For the case of D=5 generalized Chern-Simons theory, we show that this operator actually creates an open string with a topological charge at its tip. It is shown that a duality structure exists in general, relating the membrane and topological excitation operators and the corresponding dual algebra is derived. Composite topologically charged membranes are shown to possess generalized statistics that may, in particular, be fermionic. This is the first step for the bosonization procedure in these theories. Potential applications in the full quantization of 2-branes is also briefly discussed.

1) Introduction

Extended objects such as strings and membranes have been playing an important role in recent theoretical advances in high-energy physics [1] and cosmology [2]. When coupled to field theories, it happens that specific field configurations become attached to these objects, sometimes substantially influencing their physical properties. These field configurations, frequently carry a nontrivial topological charge, thereby effectively transforming the extended object into a topological excitation. Such excitations usually present the interesting feature of acting as disorder variables, being dual to their counterparts, the order variables, which can be expressed in terms of the lagrangian fields of the theory. This duality relation, physically, means that the vacuum expectation values of the corresponding operators may be used to characterize the different possible phases of the system, according to the breakdown of the corresponding symmetries. It is easy to understand the importance of the role played by topological excitations in this framework by recalling that the presence of these in the physical spectrum of excitations is closely related to the occurence of a spontaneously broken symmetry. Mathematically, duality is expressed as an algebraic relation between the creation operators associated to the order and disorder variables, the so called dual algebra, which has far reaching effects in the physical properties of the system. One possibility opened by the existence of a dual structure manifested through such an algebraic relation is the method of bosonization, by which fermions are mapped into bosons and vice-versa.

As a consequence of the presence of field configurations that become intrinsically attached to an extended object, its statistics may change from bosonic to a generalized one and, in particular, they may become fermionic. This is the statistical transmutation, first obtained for point particles in D=3 [3] an subsequently generalized for strings in D=4 [9, 10]. Bosonization, on the other hand, is a different but related procedure, by which we combine order and disorder dual variables in order to produce a new excitation with generalized statistics that may, in particular be fermionic [4, 7]. Indeed, the construction of a fermionic operator associated to a certain object, com-

pletely expressed in terms of the bosonic fields of the underlying field theory to which this object is coupled is in fact the first step towards bosonization. Subsequently, the fermionic correlation functions should be reproduced in the framework of the bosonic theory. The bosonization program has been completely implemented in a spacetime of dimension D=2 [4, 5], where it has allowed, for instance, the obtainment of exact quantum operator solutions of nonlinear theories. In D=3 it has been only partially fulfiled [6, 8].

The formulation of the above mentioned ideas both for objects of a dimension larger than one (strings) and spacetimes higher than D=4 is a natural extension of this line of investigation that may have very interesting and useful consequences. In the present work, after presenting an unified approach to the subject in dimensions D=2,3 and 4, we investigate the case of membranes and strings in a spacetime of dimension D=5, considering specifically BF and generalized Chern-Simons theories. We explicitly construct, in both cases, the creation operators for quantum excitations bearing a nonzero topological charge. In the specific case of the generalized Chern-Simons theory, we also consider the case where this excitation is an open string with a topological charge at its tip. We also explicitly obtain the creation operators of the field configurations associated to quantum membranes and show that indeed they are the eigenstates of the field operators with the correct eigenfield configurations. In both theories, we study the duality relation and derive the dual algebra satisfied by the creation operators of topologically charged states and the quantum membrane field configurations. Based on these dual algebras, we then proceed to the obtainment of composite topologically charged membranes and show that they possess general statistics that may be, in particular, fermionic. In this way, therefore, the first step in the bosonization program may be accomplished in the theories considered here. Perspectives for future extensions os this work are presented in Section 5.

2) Overview of Quantization of Topological Excitations, p-branes and Duality

2.1) Topological Charges, Magnetic Fluxes and p-branes

The identically conserved topological current is expressed in a D-dimensional spacetime in terms of a tensor field of rank D-2. In D=2,3,4 and 5, it is given, respectively by

$$J^\mu = \epsilon^{\mu\nu}\partial_\nu\phi$$

$$J^\mu = \epsilon^{\mu\alpha\beta}\partial_\alpha B_\beta$$

$$J^\mu = \epsilon^{\mu\nu\alpha\beta}\partial_\nu B_{\alpha\beta}$$

$$J^\mu = \epsilon^{\mu\lambda\nu\alpha\beta}\partial_\lambda B_{\nu\alpha\beta} \tag{2.1}$$

The generalization to higher dimensions is obvious. The topological charge, in all cases is given by

$$Q = \int d^{D-1}x J^0 \tag{2.2}$$

The magnetic field, or magnetic flux density, on the other hand, is always given in terms of the spatial components of the vector field A_μ. Indeed, we have

$$\mathcal{B} = \epsilon^{ij}\partial_i A_j$$

$$\mathcal{B}^i = \epsilon^{ijk}\partial_j A_k$$

$$\mathcal{B}^{ij} = \epsilon^{ijkl}\partial_k A_l, \tag{2.3}$$

respectively, in three, four and five spacetime dimensions. Observe that only in D=4 the magnetic field is a vector, whereas in D=2 it is not defined.

Let us consider now the current density associated to general extended objects (p-branes), namely strings, membranes or even particles, in any spacetime dimension D. For a particle (0-brane) in the point x_0, the current density is given by

$$j^\mu(x;x_0) = \int_{L(x_0)} d\xi^\mu \delta^D(x-\xi), \tag{2.4}$$

where $L(x_0)$ is the universe-line of the particle. For a string (1-brane) along the line L, we have

$$j^{\mu\nu}(x;L) = \int_{S(L)} d^2\xi^{\mu\nu}\delta^D(x-\xi), \tag{2.5}$$

where $S(L)$ is the universe-sheet of the string at L. Finally, for a membrane (2-brane) along the surface S, we have accordingly,

$$j^{\mu\alpha\beta}(x;S) = \int_{V(S)} d^3\xi^{\mu\alpha\beta}\delta^D(x-\xi), \tag{2.6}$$

where $V(S)$ is the universe-volume of the membrane at S.

In what follows, we are going to explore the interplay between the topological charge, magnetic flux and extended objects in BF and Chern-Simons theories in general. Observe that the particle, string and membrane densities associated respectively to the currents (2.4), (2.5) and (2.6) are a scalar, a vector and a rank-2 tensor, respectively j^0, j^{0i} and j^{0ij}. These is precisely the nature of the magnetic field in D=2, 3 and 4 respectively, according to (2.3). This is not a mere coincidence. It turns out that when particles, strings or membranes are coupled to generalized BF or Chern-Simons theories in D dimensions, a magnetic field (tensor in general) is imparted to the respective object.

It happens also, that a duality relation exists at the quantum level, between the magnetic field bearing object and the topological charge bearing states corresponding to (2.1)that can be constructed in the above mentioned theories.

2.2) Sequences of BF and Chern-Simons Theories

BF theories can be constructed in any spacetime dimension D, out of a vector field A_μ and a (D-2)-rank tensor field $B_{\mu_1 \dots \mu_{D-2}}$. In D=2,3, 4 and 5, respectively, we have

$$\mathcal{L}_2 = -\frac{1}{4}F^2_{\mu\nu} - \epsilon^{\mu\nu}A_\mu\partial_\nu\phi + \frac{1}{2}\partial_\mu\phi\partial^\mu\phi, \tag{2.7}$$

$$\mathcal{L}_3 = -\frac{1}{4}F^2_{\mu\nu} - \epsilon^{\mu\alpha\beta}A_\mu\partial_\alpha B_\beta - \frac{1}{4}H^2_{\mu\nu}, \tag{2.8}$$

$$\mathcal{L}_4 = -\frac{1}{4}F^2_{\mu\nu} - \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}A_\mu\partial_\nu B_{\alpha\beta} + \frac{1}{12}H^2_{\mu\alpha\beta} \tag{2.9}$$

and

$$\mathcal{L}_5 = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}\epsilon^{\mu\lambda\nu\alpha\beta}A_\mu\partial_\lambda B_{\nu\alpha\beta} + \frac{1}{24}H_{\mu\nu\alpha\beta}^2 \tag{2.10}$$

where $H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, $H_{\mu\alpha\beta} = \partial_\mu B_{\alpha\beta} +$ cyc. perm. $(\mu\alpha\beta)$ $H_{\mu\nu\alpha\beta} = \partial_\mu B_{\nu\alpha\beta} +$ cyc. perm. $(\mu\nu\alpha\beta)$ are the field intensity tensors of the B-field in D=3,4 and 5, respectively and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. A common feature of thte BF theories in any dimension is that integration over each of the fields A or B produces a mass term for the other.

The field equations obtained by varying with respect to A_μ in the three lagrangians above are

$$\partial_\nu F^{\mu\nu} = J^\mu, \tag{2.11}$$

where J^μ, the topological current, is given by each of the expressions in (2.1), respectively in D=2,3, 4 and 5. We immediately see that the topological charge is the source of the electromagnetic field in BF theories in any dimension. Varying with respect to B, in (2.8), (2.9) and (2.10), on the other hand, we get

$$\partial_\alpha H^{\alpha\mu} = \epsilon^{\mu\nu\beta}\partial_\nu A_\beta, \tag{2.12}$$

$$\partial_\alpha H^{\alpha\mu\nu} = \epsilon^{\mu\nu\alpha\beta}\partial_\alpha A_\beta, \tag{2.13}$$

$$\partial_\alpha H^{\alpha\mu\nu\lambda} = \epsilon^{\mu\nu\lambda\alpha\beta}\partial_\alpha A_\beta. \tag{2.14}$$

It is clear that the magnetic fields given by (2.3), in D=3,4 and 5, are the source of the "electric" B-field. The case of three spacetime dimensions is special because $\mathcal{B} = J^0$ and therefore magnetic flux is identical to topological charge. This fact is responsible for many of the interesting features of 3-dimensional Chern-Simons theory as we shall see below.

Generalized Chern-Simons theories may also be constructed in a spacetime of arbitrary dimension D, out of the same fields used for the construction of the BF theories above. Including the minimal coupling to particle, string and membrane densities, we have, respectively, in D=3, 4 and 5,

$$\mathcal{L}_{CS3} = \frac{1}{2}\epsilon^{\mu\alpha\beta}A_\mu\partial_\alpha A_\beta - j^\mu A_\mu, \tag{2.15}$$

$$\mathcal{L}_{CS4} = \epsilon^{\mu\nu\alpha\beta} A_\mu \partial_\nu B_{\alpha\beta} - j^{\mu\nu} B_{\mu\nu} - j^\mu A_\mu \tag{2.16}$$

and

$$\mathcal{L}_{CS5} = \epsilon^{\mu\lambda\nu\alpha\beta} A_\mu \partial_\lambda B_{\nu\alpha\beta} - j^{\nu\alpha\beta} B_{\nu\alpha\beta} - j^\mu A_\mu. \tag{2.17}$$

Notice that the case D=3 is special because, both A_μ and B_μ being vector fields, there are three possible equivalent topological terms involving AA, AB and BB, respectively. Making linear combinations of these two fields, however, it is possible to write all of them as a unique topological term as in (2.15).

Varying each of the lagrangians above with respect to A_μ, we obtain

$$j^\mu = J^\mu, \tag{2.18}$$

where J^μ is the topological current, in each case, given by Eqs. (2.1). We see that the topological charge density becomes identified with the particle density coupled to the Chern-Simons theory. In the specific case of D=3, as is well known, the topological charge density is also the magnetic flux density or magnetic field (a scalar in this case). This fact is responsible for the statistical transmutation of particles coupled to the Chern-Simons field in three dimensions [3].

If we vary (2.16) and (2.17) with respect to the B-field, on the other hand, we obtain

$$j^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} \partial_\alpha A_\beta \tag{2.19}$$

and

$$j^{\mu\nu\lambda} = \epsilon^{\mu\nu\lambda\alpha\beta} \partial_\alpha A_\beta \tag{2.20}$$

Taking the 0-component of each of the currents above, that corresponds to the string and membrane densities, respectively, we get

$$\begin{aligned} j^{0i} &= \mathcal{B}^i \\ j^{0ij} &= \mathcal{B}^{ij} \end{aligned} \tag{2.21}$$

These expressions clearly show that a magnetic field is associated to the p-brane (string or membrane) coupled to the B-field in a generalized Chern-Simons theory.

In the next subsection, we show how to construct the quantum field operators that create states bearing topological charge and also the operators creating the quantum field configuration that are eingenstates of the magnetic field operator with the eigen-field configurations corresponding to the ones associated to strings and membranes in the theories above. We also investigate the duality relation existing among them.

2.3) Creation Operators of Topological Excitations and Extended Objects

Let us examine here the general form of the topological excitation and extended object creation operators both in Chern-Simons and BF theories in arbitrary dimensions. We start with the creation operators of topological charge bearing excitations in generalized BF-theories. The best known case is D=2, where this operator is given by [4]

$$\mu(x,t) = \exp\left\{-ia\int_{-\infty}^{x} d\xi \Pi(\xi,t)\right\} \tag{2.22}$$

where $\Pi = \partial_0\phi$ is the momentum canonically conjugate to ϕ. This expression has been generalized to three and four spacetime dimensions, respectively, in the form

$$\mu(\vec{x},t) = \exp\left\{-ia\int_{-\infty}^{\vec{x}} d\xi^i \epsilon^{ij}\Pi^j(\vec{\xi},t)\right\} \tag{2.23}$$

and

$$\mu(\vec{x},t) = \exp\left\{-ia\int_{-\infty}^{\vec{x}} d\xi^i \epsilon^{ijk}\Pi^{jk}(\vec{\xi},t)\right\}, \tag{2.24}$$

where $\Pi^j = H^{oj}$ and $\Pi^{jk} = H^{ojk}$. $H^{\mu\nu}$ is the field intensity tensor of the B_μ field in D=3 and $H^{\mu\nu\alpha}$ the one of the $B_{\mu\nu}$ field in D=4. All of these three operators create eigenstates of Q, Eq. (2.2), with eigenvalue proportional to a (see [10], for D=4 and [11], for D=3). In all cases, D=2,3 and 4 above, the operator μ can be written as

$$\mu(x) = \exp\left\{-ia\int_{-\infty}^{x} dx^\mu J_\mu\right\}, \tag{2.25}$$

with J^μ given by the corresponding expression in (2.1).

Let us consider now the topological charge creation operators in generalized Chern-Simons theories given by (2.15)-(2.16). According to the results of [13, 14] the operator that creates eigenstates of the topological charge Q, Eq. (2.2) is given by the

expression

$$\mu(\vec{x},t) = \exp\left\{-ia\int_{-\infty}^{\vec{x}} d\xi^\mu A_\mu(\vec{\xi},t)\right\} \tag{2.26}$$

in both theories, respectively in D=3 and D=4. In the following sections, we are going to determine the creation operator of topological charge eigenstates, both in BF and Chern-Simons theories in D=5.

We have seen above that a magnetic field, which may be a scalar, vector or tensor, given by (2.3), is associated to a particle, string or membrane in generalized Chern-Simons theories in D=3, 4 and 5, respectively. The operator creating the quantum magnetic field eigenstates associated to these objects has been studied in [13] for the case D=3 and in [14] for D=4. For Chern-Simons theory in D=3, this operator is given by [13]

$$\sigma(x) = \exp\left\{-ib\int d^3x j^\mu A_\mu\right\}, \tag{2.27}$$

where the current j^μ is given by (2.4). It can be shown [13] that

$$\mathcal{B}|\sigma >= b|\sigma > \tag{2.28}$$

Inserting (2.4) in (2.27), we immediately find that in this case $\sigma(x) = \mu(x)$, reflecting, at the quantum level, the fact that (scalar) magnetic field and topological charge are identified in D=3 Chern-Simons theory.

For Chern-Simons theory in D=4, conversely, the operator that creates the eigenstates of the quantum vector magnetic field that is associated to a string coupled to the tensor field according to (2.16), is given by [14]

$$\sigma(L) = \exp\left\{-ib\int d^4x j^{\mu\nu}(L) B_{\mu\nu}\right\}, \tag{2.29}$$

for a string at the curve L, associated to the density $j^{\mu\nu}$ given by (2.5). It can be shown [14] that for a closed spatial string at the curve C we have, indeed,

$$\mathcal{B}^i|\sigma(C) >= 2b\left[\oint_C d\xi^i \delta(\vec{\xi}-\vec{x})\right]|\sigma(C) > . \tag{2.30}$$

Interestingly, in the case of BF theories in D=3 and D=4, the creation operator of the extended objects carrying the corresponding attached magnetic field has exactly

the same form as the ones in (2.27) and (2.29), respectively in D=3, [12] and D=4, [10, 15]. In this case they create eigenstates of the generalized "electric" B-field, which is precisely the magnetic field of A_μ, as we have seen above. Of course, in D=3 A_μ and B_μ are identified.

These observations allow us to induce that the general form of the operator creating the quantum field states associated to general extended objects, either in Chern-Simons or BF-theories, in an arbitrary dimension, should be given by

$$\sigma = \exp\left\{-ib\int d^D x j^{\mu\ldots\nu} B_{\mu\ldots\nu}\right\}, \tag{2.31}$$

where the object density $j^{\mu\ldots\nu}$ is given by (2.4), (2.5), (2.6) anf generalizations thereof. We are going to verify this explicitly fore case of membranes in D=5, in the following two sections.

2.4) Duality

There exists a remarmakable relationship between the creation operators of topological charge and extended objects in a given theory. This is order-disorder duality, in the sense of Kadanoff and Ceva [16]. Physically, the vacuum expectation value of the topological charge creation operator behaves as a disorder parameter, dual to the corresponding expectation value of the extended object creation operator. Observe that in the case of a vector field (2.31) is nothing but the Wilson loop (or line) operator. This duality has been extended and explored in continuum field theory by 't Hooft [17].

From the mathematical point of view, order-disorder duality is expressed as an algebraic relation beteen σ and μ, as first discovered in the framework of the Ising model [16], and subsequently generalized to field theory [17]. In D=2, the dual algebra is [7, 18]

$$\mu(x,t)\sigma(y,t) = \exp\left\{i\frac{ab}{2\pi}\theta(y-x)\right\}\sigma(y,t)\mu(x,t), \tag{2.32}$$

where θ is the Heaviside function. In D=3, the corresponding algebraic relation is

[12, 13]

$$\mu(\vec{x},t)\sigma(\vec{y},t) = \exp\left\{i\frac{ab}{2\pi}\arg(\vec{y}-\vec{x})\right\}\sigma(\vec{y},t)\mu(\vec{x},t). \tag{2.33}$$

The generalization of these algebraic relations existing between a point topological charge operator and the creation operator of a string at a spatial curve C for D=4 has been obtained in [10, 14] and is given by

$$\mu(\vec{x},t)\sigma(C,t) = \exp\left\{i\frac{ab}{4\pi}\Omega(\vec{y};C)\right\}\sigma(C,t)\mu(\vec{x},t), \tag{2.34}$$

where $\Omega(\vec{y};C)$ is the solid angle comprised by $\vec{y}$ and the spatial curve C.

One of the most important consequences of the dual (order-disorder) algebra is the possibility of constructing a mapping between fermionic and bosonic fields known as bosonization [4, 5]. Indeed, it follows from the above relations that the product of σ and μ fields may have generalized and, in particular, fermionic statistics. This product is precisely what appears in the bosonized expression of the Dirac field in D=2, obtained by Mandelstam [4]. Also in D=3, a similar product has been obtained in the bosonization of the free massless Dirac field [6].

3) Quantum Magnetic Membranes and Topological Excitations in the BF Theory in D=5

3.1) The Topological Charge Operator

Let us investigate here the creation operator for quantum topological charge eigenstates in the five dimensional BF theory described by the lagrangian (2.10). Following the sequence expressed in (2.22), (2.23) and (2.24), we write

$$\mu(\vec{x},t) = \exp\left\{-i\frac{a}{3}\int_{-\infty}^{\vec{x}} d\xi^i \epsilon^{ijkl}\Pi^{jkl}(\vec{\xi},t)\right\}, \tag{3.1}$$

where $\Pi^{jkl} = H^{jkl0}$ is the momentum canonically conjugate to B^{jkl}, which satisfies the following canonical commutation rules

$$[B^{ijk}(\vec{x},t),\Pi^{rst}(\vec{y},t)] = \mathrm{i}\Delta^{ijk,rst}\delta^4(\vec{x}-\vec{y}), \tag{3.2}$$

where

$$\Delta^{ijk,rst} = \Sigma_{\text{permutations }[rst]}\ (-1)^P \delta^{ir}\delta^{js}\delta^{kt}, \tag{3.3}$$

P being the parity of the permutation. These have been obtained by eliminating the second class constraints, without the use of any gauge condition.

Let us determine the commutation of μ with the topological charge operator

$$Q = \int d^4x \epsilon^{ijkl} \partial_i B_{jkl} \tag{3.4}$$

Using (3.2) and the Baker-Hausdorf formula, we readily get

$$[Q, \mu] = a\mu, \tag{3.5}$$

which implies that the states $|\mu>=\mu|0>$, created by μ are indeed eigenstates of the topological charge Q, given by (3.4), with eigenvalue a.

3.2) The Magnetic Membrane Creation Operator

Let us study here the operator creating the quantum field configuration corresponding to a magnetic membrane in D=5 BF theory. Following the sequence occurrying in (2.27), (2.29) and (2.31), we write, for a membrane at a spatial surface S,

$$\sigma(S) = \exp\left\{-ib \int d^5x j^{\mu\nu\alpha}(S) B_{\mu\nu\alpha}\right\}, \tag{3.6}$$

where $j^{\mu\nu\alpha}(S)$ is given by (2.6). Using (2.6) and performing the x integration in the above equation, we find

$$\sigma(S;t) = \exp\left\{-ib \int_{V(S)} d^3\xi^{ijk} B_{ijk}(\vec{\xi}, t)\right\}, \tag{3.7}$$

where $V(S)$ is the universe-volume of the membrane at S, which has precisely S as its border and $d^3\xi^{ijk}$, its volume element.

Let us determine the commutation relation between $\sigma(S;t)$ and the the 2-tensor magnetic field $\mathcal{B}^{ij}$, expressed by (2.3), in D=5. From (2.14), it becomes clear that in D=5 BF theory, we have

$$\mathcal{B}^{ij} = \partial_k \Pi^{kij} \tag{3.8}$$

where $\Pi^{jkl} = H^{jkl0}$ is the momentum canonically conjugate to B^{jkl}.

Let us call $\sigma(S;t) \equiv e^{\alpha(S;t)}$. Using (3.8) and (3.2), we get

$$[\mathcal{B}^{ij}(\vec{x};t), \alpha(S;t)] = b \int_{V(S)} d^3\xi^{ijk} \partial_k \delta^4(\vec{\xi} - \vec{x})$$

$$= b \oint_S d^2\xi^{ij} \delta^4(\vec{\xi} - \vec{x}), \tag{3.9}$$

where, in the last step, we used the generalized Stokes' theorem. Notice that the directions associated to i, j are tangent to the membrane at each point. Using the Baker-Hausdorf formula, we immediately find

$$[\mathcal{B}^{ij}(\vec{x};t), \sigma(S;t)] = b \left[\oint_S d^2\xi^{ij} \delta^4(\vec{\xi} - \vec{x}) \right] \sigma(S;t), \tag{3.10}$$

which implies

$$\mathcal{B}^{ij}(\vec{x};t)|\sigma(S;t) >= b \left[\oint_S d^2\xi^{ij} \delta^4(\vec{\xi} - \vec{x}) \right] |\sigma(S;t) > . \tag{3.11}$$

This shows that $\sigma(S;t)$ creates eigenstates of the magnetic field with an eigenconfiguration that is nonvanishing along the membrane. Integrating the magnetic field along a surface R orthogonal to the membrane, we get the magnetic flux

$$\Phi_R = \int_R d^2\eta^{ij} \mathcal{B}^{ij} \tag{3.12}$$

satisfying

$$\Phi_R|\sigma(S;t) >= b|\sigma(S;t) > . \tag{3.13}$$

3.3) Duality, Generalized Statistics and Fermionization

Let us show now that a duality relation that generalizes (2.32), (2.33) and (2.34) can be established between the topological charge creation operator μ, given by (3.1) and the membrane operator σ, given by (3.7), in the framework of BF theory in five spacetime dimensions. Indeed, calling $\mu(\vec{x};t) \equiv e^{\beta(\vec{x};t)}$ and again $\sigma(S;t) \equiv e^{\alpha(S;t)}$, we obtain, using (3.2),

$$[\beta(\vec{x};t), \alpha(S;t)] = -\mathrm{i}ab \int_{V(S)} d^3\eta^i \int_{-\infty}^{\vec{x}} d\xi^i \delta^4(\vec{\xi} - \vec{\eta}), \tag{3.14}$$

where $d^3\eta^i$ is the volume element of $V(S)$. Using the result (6.4) of the Appendix, we get

$$
\begin{aligned}
[\beta(\vec{x};t),\alpha(S;t)] &= \mathrm{i}\frac{12ab}{4\pi^2}\int_{V(S)} d^3\eta^i \frac{(\vec{x}-\vec{\eta})^i}{|\vec{x}-\vec{\eta}|^4} \\
&= \mathrm{i}\frac{12ab}{4\pi^2}\Omega_3(\vec{x};S),
\end{aligned}
\tag{3.15}
$$

where $\Omega_3(\vec{x};S)$ is the hypersolid angle comprised by the point $\vec{x}$ and the surface S associated to the membrane. This is defined by $dV = R^3 d\Omega_3$, where dV is the element of the volume enclosed by the surface S.

Since the above commutator is a c-number, using the Baker-Hausdorf formula, we readily find

$$
\mu(\vec{x};t)\sigma(S;t) = \exp\left\{\mathrm{i}\frac{12ab}{4\pi^2}\Omega_3(\vec{x};S)\right\}\sigma(S;t)\mu(\vec{x};t) \tag{3.16}
$$

This is the dual algebra satisfied by the topological excitation and membrane creation operators in D=5 BF theory, that generalizes the corresponding expressions in D=2,3 and 4, given respectively by (2.32), (2.33) and (2.34).

We now construct the creation operator for the composite state carrying both topological charge and tensor magnetic flux along a closed membrane along the surface S. We choose S_x to be a sphere of radius R centered at $\vec{x}$ and place the topological charge in the center, namely,

$$
\psi(x;S_x;t) = \lim_{\vec{x}\to\vec{y}} \mu(\vec{x},t)\sigma(S_y,t) \tag{3.17}
$$

Using the fact that $\Omega_3(\vec{x};S_y) - \Omega_3(\vec{y};S_x) = 4\pi^2\epsilon\left(\Omega_3(\vec{x};S_y)\right)$, where $\epsilon(x)$ is the sign function, we obtain from (3.16)

$$
\psi(x;S_x;t)\psi(y;S_y;t) = e^{i\ 12ab\epsilon(\Omega_3(\vec{x};C_y))}\psi(y;S_y;t)\psi(x;S_x;t). \tag{3.18}
$$

This relation shows that a topologically charged membrane, created by the operator ψ, defined in (3.17), possesses generalized statistics determined by the parameters a and b. For a suitable choice of the parameters a and b, we can obtain fermionic states even though we are in the framework of a purely bosonic BF theory.

4) Bosonic Membranes and Strings Coupled to the Generalized Chern-Simons Theory in D=5

4.1) The Topological Charge Operator

Let us consider in this section the generalized Chern-Simons theory coupled to a membrane and an external source j^μ, described by the lagrangian given by (2.17). We shall firstly study the topological excitations creation operator. Inspired in (2.26), we write

$$\mu(\vec{x},t) = \exp\left\{-i\frac{a}{3}\int_{-\infty,L}^{\vec{x}} d\xi^i A^i(\vec{\xi},t)\right\}, \tag{4.1}$$

Let us evaluate the commutation relation of this operator with the topological charge Q, which in the present theory is also given by (3.4). For this, we need the canonical commutation rules for the D=5 Chern-Simons theory, namely

$$[B^{ijk}(\vec{x},t), A^l(\vec{y},t)] = \mathrm{i}\epsilon^{ijkl}\delta^4(\vec{x}-\vec{y})$$

$$[B^{ijk}(\vec{x},t), B^{lmn}(\vec{y},t)] = [A^i(\vec{x},t), A^j(\vec{y},t)] = 0, \tag{4.2}$$

which, again, have been obtained by eliminating the second class constraints without fixing any gauge condition. Calling $\mu(\vec{x},t) \equiv e^{\beta(\vec{x},t)}$ and using (4.2), we find

$$[Q, \beta(\vec{x},t)] = a\int d^4y \int_{-\infty}^{\vec{x}} d\xi^i \partial^i_{(\xi)}\delta^4(\vec{\xi}-\vec{y}) = a \tag{4.3}$$

Since this is a c-number, we immediately conclude that

$$[Q, \mu] = a\mu, \tag{4.4}$$

implying that the operator given by (4.1) indeed creates topologically charged excitations, namely eigenstates of Q.

A very interesting special case of the theory described by (2.17) is the one when the source j^μ is given by

$$j^\mu = \oint_C d\xi^\mu \delta^5(x-\xi), \tag{4.5}$$

where C is the border of the universe-sheet of a string. For a closed spatial string, we have $j^0 = 0$, implying, according to (2.18), that this type of string does not bear

any topological charge. For an open spatial string, however, $j^0 \neq 0$ and we see that it carries a nonvanishing topological charge. Notice that, in this case, C is still closed despite of the fact that the string is open. We conclude that, in this situation, the operator $\mu(\vec{x}, t)$ in (4.1) creates an open string with extremity at the point $\vec{x}$, along the line L.

4.2) The Membrane Creation Operator

Let us consider now the operator creating the quantum field configuration associated to the membrane coupled to the generalized Chern-Simons theory in D=5. From (2.20) and (2.21), we see that there is a 2-tensor magnetic field attached to the membrane coupled to the tensor field in (2.17). Following (2.31) we write the membrane creation operator in the present case as $\sigma(S;t) \equiv e^{\alpha(S;t)}$ where $\sigma(S;t)$ is given by expressions identical to (3.6) and (3.7), which we had in D=5 BF theory. Using (2.3) and the canonical commutation rules (4.2), we find that expressions (3.9), (3.10),(3.11) and (3.13) also hold here. These imply that also here the operator $\sigma(S;t)$ creates the correct eigenstates of the magnetic field attached to the quantum membrane.

4.3) Duality, Generalized Statistics and Fermionization

Using the canonical commutation rules (4.2), and following the same steps as in the previous section, it is easy to see that the topological excitation operator $\mu(\vec{x}, t)$, given by (4.1) and the membrane operator $\sigma(S;t)$, given by (3.7) satisfy precisely the same relations expressed in (3.14) to (3.18). In particular the dual algebra (3.16) is the same. As a consequece, also the composite states of a membrane and a point topological charge have generalized statistics that may be, in particular, fermionic. Of course the previous remarks also hold when μ may be regarded as the creation operator of an open string.

5) Perspectives

The general duality structure investigated here has proved to be completely general and can be extended for any spacetime dimension. It may also be applied to extended objects of any dimensionality. The formalism introduced here shall certainly be required whenever a quantum description of extended objects such as strings or membranes coupled to a field theory should be needed. The full quantization of an extended object (p-brane) is in general plagued by difficulties, usually associated with anomalies. These, in a light-cone gauge formulation, would appear in the algebra of generators of the Lorentz group. Only superstrings and the supermembrane in D=11 are known to be free of these anomalies [19]. Nonsupersymmetric p-branes, on the other hand, are known to suffer from conformal anomalies in general. The absence of supersymmetry anomalies is associated to the presence of massless states in the spectrum [20]. It is precisely in connection to this fact that the present formulation is potentially useful for the full quantization of p-branes. Indeed, dual order-disorder algebras such as (3.16), (2.32), (2.33) and (2.34) have been shown [21] to be closely related to a criterion for the presence of massless states in the spectrum. This actually happens whenever the expectation values of both operators satisfying the dual algebra vanish.

The natural extension of this work involves the evaluation of correlation functions of membrane and topological excitation creation operators in BF and Chern-Simons theories, both in D=5 and D=4. This study will lead, among other things, to a detailed description of the quantum spectrum of excitations for these extended objects as well as of their scattering properties. It shall also certainly lead to very interesting unfoldings regarding bosonization in higher dimensions.

6) Appendix

In a spacetime of dimension D=4, the spatial laplacian obeys the equation

$$-\nabla^2 \left[\frac{1}{4\pi|\vec{\xi} - \vec{x}|} \right] = \delta^3(\vec{\xi} - \vec{x}) \tag{6.1}$$

and the above Green's function satisfies the identity

$$\partial^i_{(\xi)} \left[\frac{1}{4\pi|\vec{\xi} - \vec{x}|} \right] = \begin{cases} \epsilon^{ijk} \partial^j_{(\xi)} \varphi^k(\vec{\xi} - \vec{x}); & \vec{\xi} \notin V_L \\ \int_{-\infty,L}^{\vec{x}} d\eta^i \delta^3(\vec{\eta} - \vec{\xi}); & \vec{\xi} \in V_L \end{cases} , \tag{6.2}$$

where V_L is a cone of infinitesimal angle with vertex at $\vec{\xi} = \vec{x}$ and axis along the line $L : (-\infty, \vec{x})$ and $\vec{\varphi} = \frac{1-\cos\theta}{r\sin\theta}\hat{\varphi}$, with $r = |\vec{\xi} - \vec{x}|$.

In a spacetime of dimension D=5, conversely, we have for the spatial laplacian

$$-\nabla^2 \left[\frac{1}{4\pi^2|\vec{\xi} - \vec{x}|^2} \right] = \delta^4(\vec{\xi} - \vec{x}) \tag{6.3}$$

Inside a hypercone containing L and with the tip in $\vec{x}$ we have, accordingly, the identity

$$\partial^i_{(\xi)} \left[\frac{1}{4\pi^2|\vec{\xi} - \vec{x}|^2} \right] = \int_{-\infty,L}^{\vec{x}} d\eta^i \delta^4(\vec{\eta} - \vec{\xi}), \tag{6.4}$$

which can be easily verified by applying $\partial^i_{(\xi)}$. Observe now that, because of the δ-function, (3.14) is non vanishing only inside V_L, hence we can use (6.4), in order to establish (3.15).

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